ONE-WAY ANOVA

Q1) Define the one-way ANOVA model

$$y_{ij} = \mu + \tau_i + \varepsilon_{ij}, \quad \varepsilon_{ij} \overset{iid}{\sim} (0, \sigma^2) \quad (i = 1, \cdots, k) \,|\, j = 1, \cdots, n_i)$$

μ: overall mean without the treatment effect

τ_i: Contribution of the ith treatment to the mean

Q2) What does "one-way" mean in the one-way ANOVA model?

One-way refers to the fact that the number of treatment type is one.

Q3) What would a "two-way" ANOVA model mean?

It refers to the fact that the model considers two treatment types

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Consider the following one-way ANOVA model:

$$y_{ij} = \mu + \tau_i + \varepsilon_{ij}, \quad \varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2), \quad i = 1, 2 \quad j = 1, 2, 3$$

$y_{11} = \mu + \tau_1 + \varepsilon_{11}$ $\quad y_{12} = \mu + \tau_1 + \varepsilon_{12}$ $\quad y_{13} = \mu + \tau_1 + \varepsilon_{13}$

$y_{21} = \mu + \tau_2 + \varepsilon_{21}$ $\quad y_{22} = \mu + \tau_2 + \varepsilon_{22}$ $\quad y_{23} = \mu + \tau_2 + \varepsilon_{23}$

$$y = \begin{bmatrix} y_{11} \\ y_{12} \\ y_{13} \\ y_{21} \\ y_{22} \\ y_{23} \end{bmatrix} \quad X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad \beta = \begin{bmatrix} \mu \\ \tau_1 \\ \tau_2 \end{bmatrix} \quad \varepsilon = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{21} \\ \varepsilon_{22} \\ \varepsilon_{23} \end{bmatrix}$$

Q4) In the one-way ANOVA model, the design matrix is NOT full rank. What kind of problem would arise if we were to fit a regression model.

The matrix X^TX is singular and thus non-invertible.

If the design matrix X is not full-rank, the ANOVA model is not identifiable

Q5) What constraint is imposed so that the design matrix X is full rank?

Last column of the design matrix is eliminated ($\because \mu_k = 0$)

Q6) Prove Theorem 3

Consider the one-way ANOVA model with constraint $\tau_k = 0$

$$y_{ij} = \mu + \tau_i + \varepsilon_{ij}, \quad \varepsilon_{ij} \overset{i.i.d}{\sim} N(0, \sigma^2) \quad (i = 1, 2, \cdots, n_i)$$

In this case, the least squares estimators for $\mu, \tau_1 \cdots \tau_{k-1}$ are given by:

$$\hat{\mu} = \bar{y}_k \quad \text{and} \quad \hat{\tau}_i = \bar{y}_i - \bar{y}_k \quad (i = 1 \cdots k-1)$$

$$y = \begin{bmatrix} y_{11} \\ \vdots \\ y_{1n_1} \\ \vdots \\ y_{k1} \\ \vdots \\ y_{kn_k} \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} \mathbb{1}_{n_1} & \mathbb{1}_{n_1} & 0 & \cdots & 0 \\ \mathbb{1}_{n_2} & 0 & \mathbb{1}_{n_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \mathbb{1}_{n_{k-1}} & 0 & 0 & \cdots & \mathbb{1}_{n_{k-1}} \\ \mathbb{1}_{n_k} & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\beta = \begin{bmatrix} \mu \\ \tau_1 \\ \vdots \\ \tau_{k-1} \end{bmatrix} \quad \text{and} \quad \varepsilon = \begin{bmatrix} \varepsilon_{11} \\ \vdots \\ \varepsilon_{1n_1} \\ \vdots \\ \varepsilon_{k1} \\ \vdots \\ \varepsilon_{kn_k} \end{bmatrix}$$

$$\hat{\beta} = (X^TX)^{-1}X^Ty \Leftrightarrow (X^TX)\hat{\beta} = X^Ty$$

$$X^TX = \begin{bmatrix} n & n_1 & n_2 & \cdots & n_{k-1} \\ n_1 & n_1 & 0 & \cdots & 0 \\ n_2 & 0 & n_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n_{k-1} & 0 & 0 & \cdots & n_{k-1} \end{bmatrix}, \quad X^Ty = \begin{bmatrix} y_{\cdot} \\ y_{1\cdot} \\ \vdots \\ y_{k-1\cdot} \\ 0 \end{bmatrix}, \quad y_{\cdot} = \frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{n_i} y_{ij}$$

$$y_{i\cdot} = \frac{1}{n_i}\sum_{j=1}^{n_i} y_{ij}$$

$$(X^TX)\hat{\beta} = \begin{bmatrix} n & n_1 & n_2 & \cdots & n_{k-1} \\ n_1 & n_1 & 0 & \cdots & 0 \\ n_2 & 0 & n_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n_{k-1} & 0 & 0 & \cdots & n_{k-1} \end{bmatrix} \begin{bmatrix} \mu \\ \tau_1 \\ \vdots \\ \tau_{k-1} \end{bmatrix} = \begin{bmatrix} n\mu + n_1\tau_1 + \cdots + n_{k-1}\tau_{k-1} \\ n_1\mu + n_1\tau_1 \\ n_2\mu + n_2\tau_2 \\ \vdots \\ n_{k-1}\mu + n_{k-1}\tau_{k-1} \end{bmatrix}$$

$$(n\mu + n_1\tau_1 + \cdots + n_{k-1}\tau_{k-1}) - (n_1\mu + n_1\tau_1) - \cdots - (n_{k-1}\mu + n_{k-1}\tau_{k-1})$$

$$= (n - n_1 - \cdots - n_{k-1})\mu$$

$$= y_{\cdot} - (y_{1\cdot} + \cdots + y_{k-1\cdot}) = y_k$$

$$\therefore n_k\mu = y_k \quad \therefore \hat{\mu} = \bar{y}_k$$

$$n_i\hat{\mu} + n_i\hat{\tau}_i = y_{i\cdot} \quad \therefore \tau_i = \bar{y}_i - \bar{y}_k \quad (i = 1, \cdots, k-1)$$

Q7) Consider the one-way ANOVA model with constraint $\tau_K = 0$ and normality assumption. We wish to test whether there is treatment effect or not.

$$H_0: \tau_1 = \cdots = \tau_{K-1} = 0 \quad \text{vs} \quad H_1: \text{not } H_0$$

Consider the test statistic F defined as below:

$$F = \frac{SSB/(K-1)}{SSW/(n-K)}, \quad SSB = \sum_{i=1}^{K} n_i(\bar{y}_i - \bar{y})^2 \text{ and } SSW = \sum_{i=1}^{K}\sum_{j=1}^{n_i}(y_{ij} - \bar{y}_i)^2$$

Show that under H_0: $F \sim F(K-1, n-K)$

$$SSB = \sum_{i=1}^{K} n_i(\bar{y}_i - \bar{y})^2 = \|(\Pi_p - \Pi_{\mathbb{1}})y\|^2 = y^T(\Pi_p - \Pi_{\mathbb{1}})y, \quad \Pi_p = X_p(X_p^T X_p)^{-1}X_p^T$$

$$X_p = \begin{bmatrix} \mathbb{1}_{n_1} & 0 & \cdots & 0 \\ 0 & \mathbb{1}_{n_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbb{1}_{n_k} \end{bmatrix}$$

$$SSB/\sigma^2 = y^T\left(\frac{\Pi_p - \Pi_{\mathbb{1}}}{\sigma^2}\right)y = y^T A y$$

$$A\Sigma = \left(\frac{\Pi_p - \Pi_{\mathbb{1}}}{\sigma^2}\right)\sigma^2 I_n = \Pi_p - \Pi_{\mathbb{1}} : \text{idempotent}$$

$$r = \text{rank}(A) = \text{trace}(\Pi_p) - \text{trace}(\Pi_{\mathbb{1}}) = K - 1$$

$$\delta = E[y]^T A E[y] = 0 \quad \because E[y] = 0 \text{ under } H_0$$

$$\therefore SSB/\sigma^2 \sim \chi^2(K-1)$$

$$SSW/\sigma^2 = y^T\left(\frac{I_n - \Pi_p}{\sigma^2}\right)y = y^T B y$$

$$B\Sigma = \left(\frac{I_n - \Pi_p}{\sigma^2}\right)\sigma^2 I_n = \Pi_p - \Pi_{\mathbb{1}} : \text{idempotent}$$

$$r = \text{rank}(B) = \text{trace}(I_n) - \text{trace}(\Pi_p) = n - K$$

$$\delta = E[y]^T B E[y] = 0 \quad \because E[y] = 0 \text{ under } H_0$$

$$SSW/\sigma^2 \sim \chi^2(n-K)$$

$$SSB \perp SSB \quad \because A\Sigma B = (\Pi_p - \Pi_{\mathbb{1}})\sigma^2 I_n (I_n - \Pi_p)$$
$$= \sigma^2(\Pi_p - \Pi_p - \Pi_{\mathbb{1}} + \Pi_{\mathbb{1}}) = 0$$

$$\therefore \frac{SSB/(K-1)}{SSW/(n-K)} \sim F(K-1, n-K)$$